REMARKS on terminology

6.24

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$$\Sigma^n \subset (M^{n+k}, g)$$

$(e_\mu{}^A, n_a^B)$ – orthonormal frame on Σ^n

$$de^A_\mu = b_{\mu a} n^A{}_a + \gamma_{\nu\mu} e_\nu{}^A.$$

$$b_{\mu a} = b_{\mu\nu a}\theta^\nu \qquad b_{\mu\nu a} = b_{\nu\mu a}$$

$$\Rightarrow \quad \boxed{B^A = b_{\mu\nu a}\theta^\mu\theta^\nu \cdot n_a^A} \quad \text{second fundamental form.}$$

$\eta \in (T\Sigma)^\perp$, $X, Y \in T\Sigma$ $\qquad X = X^\mu e_\mu \quad Y = Y^\nu e_\nu$, $\quad \eta = N_a n_a$

$$H_\eta(X,Y) = g(B(X,Y),\eta) = g_{AB} B^A(X,Y)\eta^B =$$

$$= g_{AB}\, b_{\mu\nu a} X^\mu Y^\nu n_a^A N_b n_b^B =$$

$$= b_{\mu\nu a} X^\mu Y^\nu N_a =$$

$$= (b_{\mu\nu a} Y^\nu N_a) X^\mu$$

$\cancel{g(S_\eta(X),Y)}$

~~Observe that S_η defined on T_pM by~~

~~$S_\eta(X)$~~

Define

$$S_\eta : T_pM \longrightarrow TM \quad \text{by:}$$

$$[S_\eta(X)]_\mu = b_{\mu\nu a} X^\nu N_a e_\mu$$

$$g(S_\eta(X),Y) = H_\eta(X,Y) = g(S_\eta(Y),X)$$

S_η is a symetric operator on $T_p\Sigma$ for each $p \in \Sigma$

There exists an orthonormal basis in $T_p\Sigma$ s.t.

$$S_\eta(e_\mu) = \lambda_\mu e_\mu \qquad \lambda_\mu \text{ - real eigenvalues.}$$

no summation

(e_μ, n_a) and e_μ diagonalize S_η.

If $|\eta| = 1$ and Σ^n is hypersurface $(k=1)$ then we can take (e_μ, η) and η is unique if we want (e_μ, η) to agree with the orientation of T_pM

e_μ - are called principal directions

λ_μ - are called principal curvatures

invariants of the immersion

$\det S_\eta = \lambda_1 \dots \lambda_n$ Gauss-Kronecher curvature

$\frac{1}{n} Tr\, S_n = \frac{\lambda_1 + \dots + \lambda_n}{n}$ mean curvature

~~Easy to see that if all λ_μ distinct then e_μ are unique.~~

Isoparametric hypersurfaces

$\Sigma^n \subset (M^{n+1}, g)$, where (M^{n+1}, g) is a space of constant curvature, is called ISOPARAMETRIC iff all its principal curvatures are constant.

Results:

1) $\boxed{M^{n+1} = \mathbb{R}^{n+1}} \Rightarrow \Sigma$ has at most $\boxed{\text{two}}$ distinct principal curvatures
isoparametric

and must be an open subset of

a) hyperplane
or
b) hypersphere
or
c) spherical cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$

Levi-Civita for $n+1=3$ 1937
Segre for arbitrary n. 1938

2) $M^{n+1} = \mathbb{H}^{n+1} \Rightarrow$ isoparametric Σ has at most 2 distinct principal curvatures

~~and~~ must be either $\left\{ \begin{array}{l} \text{open subset of } \mathbb{S}^k \times \mathbb{H}^{n-k} \\ \text{or} \quad \text{be totally umbilic.} \end{array} \right\}$

Cartan 1938

3) $M^{n+1} = \mathbb{S}^{n+1}$ more interesting situation!

Cartan 1938 found isoparametric $\Sigma^n \subset \mathbb{S}^{n+1}$

with 1, 2, 3 and 4 distinct principal curvatures.

Münzner : number g of distinct principal curvatures of an isoparametric hypersurface $\Sigma^n \subset \mathbb{S}^{n+1}$ can be 1, 2, 3, 4 or 6.

$g \leq 3$ Cartan:

$g=1 \Rightarrow \Sigma$ is a great or small sphere in S^{n+1}

$g=2 \Rightarrow \Sigma$ is a standard product of two spheres

$$S^k(r) \times S^{n-k}(s) \subset S^{n+1}$$

$g=3 \Rightarrow$ all the principal curvatures have to have the same multiplicity 1, 2, 4, or 8.

$g=6 \Rightarrow$ all have the same multiplicity $m=1$ or 2,

Cartan / <u>Münzner</u>:

Isoparametric hypersurface in S^{n+1} is given in terms of ~~as a level surface of a~~ a ~~function~~ polynomial

$$F : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$$

of degree g satisfying

$$|\nabla F|^2 = g^2 \left(x^{1\,2} + \dots + x^{n+2\,2}\right)^{g-1}$$

$$\Delta F = \frac{m_2 - m_1}{2} g^2 \left(x^{1\,2} + \dots + x^{n+2\,2}\right)^{\frac{g}{2}-1}$$

$g=4$
Cartan examples
with $m=1$ in S^5
$m=2$ in S^9

where m_1 and m_2 are multiplicities of principal curvatures, which either are all equal or there are only two different multiplicites.

Then $\Sigma^n = \left\{ x^i \in \mathbb{R}^{n+2} \text{ s.t. } \begin{array}{l} F = \text{const} \\ x^{1\,2} + \dots + x^{n+2\,2} = 1 \end{array} \right\}$

Cartan

$$g=3 \Rightarrow \begin{cases} 1) & |\nabla F|^2 = 9\left(x^{1\,2}+\dots+x^{n+2\,2}\right)^2 \\ 2) & \Delta F = 0 \end{cases}, \qquad m_1=m_2$$

$$F = F_{\mu\nu\rho}\, x^\mu x^\nu x^\rho$$

$$\nabla_\mu F = 3 F_{\mu\nu\rho}\, x^\nu x^\rho$$

$$|\nabla F|^2 = 9\, F_{\mu\nu\rho}\, x^\nu x^\rho\, F_{\mu\alpha\beta}\, x^\alpha x^\beta$$

$$\shortparallel$$

$$9\quad g_{\nu\rho}\, x^\nu x^\rho\, g_{\alpha\beta}\, x^\alpha x^\beta$$

$$\Rightarrow \boxed{g^{\mu\rho} F_{\mu(\nu\rho} F_{\alpha\beta)\rho} = g_{(\nu\rho}\, g_{\alpha\beta)}} \quad 1)$$

$$\boxed{g^{\mu\nu} F_{\mu\nu\rho} = 0} \quad 2)$$

What are the dimensions $n+2$ in which a symmetric tensor with properties 1) and 2) exist?

Cartan

$n+2 = 5, 8, 14, 26.$

dim 5: $A \in M_{3\times 3}(\mathbb{R})$ s.t. $A^T = A$, $\operatorname{Tr} A = 0$

Space of such matrices is a 5-dim. vector space $\simeq \mathbb{R}^5$

$$A = \begin{pmatrix} x^5 - \sqrt{3}x^4 & \sqrt{3}x^3 & \sqrt{3}x^2 \\ & x^5+\sqrt{3}x^4 & \sqrt{3}x^1 \\ & & -2x^5 \end{pmatrix} \Rightarrow F = \tfrac{1}{2}\det A$$

Satisfies 1), 2) with $g=3$.

Why 5, 8, 14, 26 ?

Because

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Take $A \in M_{3\times 3}(\mathbb{K})$ s.t. $A^{+} = A$, $\operatorname{Tr} A = 0$

$$n = 2 + 3 \cdot \begin{cases} 1 \\ 2 \\ 4 \\ 8 \end{cases}$$

$$F = \frac{1}{2} \det A$$

Problem define det for $A \in M_{3\times 3}(\mathbb{H})$, $M_{3\times 3}(\mathbb{O})$.

$n =$ as above

Define: $G \subset SO(n)$ by

$$G \ni a \iff F(ax, ax, ax) = F(x, x, x).$$

Check that

$$\begin{array}{lcccc} G = & SO(3) & SU(3) & Sp(3) & F_4 \\ n = & 5 & 8 & 14 & 26 \end{array}$$

each group being in a n dimensional irreducible representation

this in particular means that

$SO(3)$ seats in a nonstandard way in $SO(5)$